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Thermal boundary-layer theory near the stagnation point in three-dimensional fluctuating flow

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The equations of motion and energy governing a three-dimensional fluctuating flow of an incompressible fluid in the vicinity of a stagnation point on a regular surface have been integrated analytically. The velocity of the oncoming flow relative to the body oscillates in magnitude but not in direction.

It has also been shown that the analysis of Lighthill for the two-dimensional fluctuating flow may be extended to the three-dimensional flow (both chordwise and spanwise), namely for each point on the body there is a critical frequency ω_0 such that for frequencies $\omega > \omega_0$ the oscillations are to a close approximation ordinary 'shear waves', unaffected by the mean flow; the phase advance in the skin friction is then 45°. For frequencies $\omega < \omega_0$ the oscillations may be closely approximated by the sum of two parts: one quasi-steady part and the other proportional to the acceleration of the oncoming stream. The phase advance in the skin friction is then $\tan^{-1}(\omega/\omega_0)$.

1. Introduction

In connexion with flutter problems, attention has been drawn to the problem of a laminar boundary layer in which the mainstream velocity fluctuates in magnitude and direction. Lighthill (1954) studied the laminar boundary-layer flow of an incompressible fluid on an infinite cylinder with fluctuation in magnitude of the mainstream velocity. Subsequently, similar problems were studied by Stuart (1955), Glauert (1956), Watson (1958, 1959), Ghoshal (1966), and others. In the present paper an attempt has been made to study the three-dimensional heat-conducting fluctuating flow near the stagnation point. Meksyn's (1956) asymptotic method has been extended to integrate the equations of motion and energy for three-dimensional fluctuating flow near the stagnation point on a regular surface. To get a physical idea of the results, they have been approximated employing Lighthill's (1954) method. For large frequency his analysis is employed to integrate the equations of motion and energy. Expressions for heat transfer and skin friction are obtained and compared for different configurations of the body.

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2. Basic equations and analysis

We consider here a fluctuating three-dimensional laminar flow of an incompressible fluid near the stagnation point of a regular surface. Dissipation effects are assumed to be small and the fluid properties to be invariable. The boundary-layer equations for the flow with reference to a body-oriented orthogonal system of co-ordinates are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial W}{\partial \zeta}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+W\frac{\partial u}{\partial \zeta}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+\nu\frac{\partial^2 u}{\partial \zeta^2}, \tag{2}$$

$$\frac{\partial V}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+W\frac{\partial v}{\partial \zeta}=\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+\nu\frac{\partial^2 v}{\partial \zeta^2}, \tag{3}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+W\frac{\partial T}{\partial \zeta}=\frac{\nu}{P_r}\frac{\partial^2 T}{\partial \zeta^2}, \tag{4}$$

where the axes OX and OY are chosen along the irrotational velocity components $U=U_0(1+\epsilon e^{i\omega t})$ and $V=V_0(1+\epsilon e^{i\omega t})$ (see Howarth 1951), ζ along the local normal ($U_0=ax$, $V_0=by$ here). The boundary conditions are

$$\left.\begin{array}{l} u(x,y,0,t)=v(x,y,0,t)=W(x,y,0,t)=0 \quad \text{and} \quad T(x,y,0,t)=T_\omega, \\ \lim\limits_{\zeta\to\infty} u(x,y,\zeta,t)=U, \quad \lim\limits_{\zeta\to\infty} v(x,y,\zeta,t)=V, \quad \lim\limits_{\zeta\to\infty} T(x,y,\zeta,t)=T_\infty. \end{array}\right\} \tag{5}$$

Following Lighthill (1954), we assume a set of solutions of the form

$$u=ax[f_0'(z)+\epsilon e^{i\omega t}f_1'(z)], \tag{6}$$

$$v=ay[g_0'(z)+\epsilon e^{i\omega t}g_1'(z)], \tag{7}$$

$$T=T_0+\epsilon T_1 e^{i\omega t}, \tag{8}$$

where ϵ is small and $z=(a/\nu)^{\frac{1}{2}}\zeta$, $f_0'=df_0/dz$, $g_0'=dg_0/dz$, etc. Now (1) with the boundary conditions (5) gives

$$W=-(\nu/a)^{\frac{1}{2}}[(af_0+bg_0)+\epsilon e^{i\omega t}(af_1+bg_1)]. \tag{9}$$

Let
$$\theta=\frac{T-T_\infty}{T_\omega-T_\infty}=\frac{T_0-T_\infty}{T_\omega-T_\infty}+\epsilon\frac{T_1}{T_\omega-T_\infty}e^{i\omega t}=\theta_0+\epsilon\theta_1 e^{i\omega t}. \tag{10}$$

Substituting these quantities in (2), (3) and (4), and equating the terms independent of ϵ, and first order in ϵ from both the sides, we obtain,

$$\left.\begin{array}{l} f_0'''+(f_0+cg_0)f_0''=f_0'^2-1, \\ g_0'''+(f_0+cg_0)g_0''=c(g_0'^2-1), \\ \theta_0''+P_r(f_0+cg_0)\theta_0'=0, \end{array}\right\} \tag{11}$$

$$\left.\begin{array}{l} f_1'''+[(f_0+cg_0)f_1''+(f_1+cg_1)f_0'']=2(f_0'f_1'-1)+i\lambda(f_1'-1), \\ g_1'''+[(f_0+cg_0)g_1''+(f_1+cg_1)g_0'']=2c(g_0'g_1'-1)+i\lambda(g_1'-1), \\ \theta_1''+P_r[\theta_1'(f_0+cg_0)+\theta_0'(f_1+cg_1)]=P_r i\lambda\theta_1, \end{array}\right\} \tag{12}$$

where $c = b/a$ and $\omega/a = \lambda$ $(0 \leqslant c \leqslant 1)$. The last equations of (11) and (12) imply $\theta = \theta(z)$ only. The boundary conditions are

$$\left.\begin{aligned} &f_0(0) = f_0'(0) = g_0(0) = g_0'(0) = 0, \quad \theta_0(0) = 1, \\ &\lim_{z\to\infty} f_0'(z) = 1, \quad \lim_{z\to\infty} g_0'(z) = 1, \quad \lim_{z\to\infty} \theta_0(z) = 0, \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} &f_1(0) = f_1'(0) = g_1(0) = g_1'(0) = 0, \quad \theta_1(0) = 0, \\ &\lim_{z\to\infty} f_1'(z) = 1, \quad \lim_{z\to\infty} g_1'(z) = 1, \quad \lim_{z\to\infty} \theta_1(z) = 0. \end{aligned}\right\} \tag{14}$$

The steady equations (11) with the boundary conditions (13) have been discussed before by many, namely Howarth (1951), Hayday & Bowlus (1967). Also of interest here is a study of the steady boundary-layer flow at a saddle point given by Davey (1961).

If $c = b = 0$ the equations for steady and fluctuating flow reduce to the equations for the corresponding two-dimensional flow (Lighthill 1954).

But we get a different type of flow if we take the limit in (11) and (12) as $c \to 0$. The resulting equations are

$$\left.\begin{aligned} f_0''' + f_0 f_0'' &= f_0'^2 - 1, \\ g_0''' + f_0 g_0'' &= 0, \\ \theta_0'' + P_r f_0 \theta_0' &= 0, \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} f_1''' + f_0 f_1'' + f_1 f_0'' &= 2(f_0' f_1 - 1) + i\lambda(f_1' - 1), \\ g_1''' + f_0 g_1'' + f_1 g_0'' &= i\lambda(g_1' - 1), \\ \theta_1'' + P_r[\theta_1' f_0 + \theta_0' f_1] &= P_r i\lambda\theta_1. \end{aligned}\right\} \tag{16}$$

It is well known that the equations (15) govern the flow near the stagnation point on a circular cylinder, unbounded in the Y direction with its axis inclined at an angle $\alpha = \tan^{-1}(V_0|U_0)$ to the mainstream. In this case V is $V = V_0(1 + \epsilon\, e^{i\omega t})$, where $V_0 =$ constant. It is also evident that the chordwise flow is unaffected by the spanwise (g flow) motion in the steady case. The situation illustrates the 'independence principle', which was applied by Sears (1948) and Gortler (1952).

From (16) it is clear that this 'independent principle' is also valid in the case of fluctuating flow.

3. Integration of the equations

To integrate the equations we shall employ a procedure of the type given by Meksyn (1956) and also used by Hayday & Bowlus (1967). The functions $f_1(z)$ and $g_1(z)$ are given by the series

$$f_1(z) = \sum_{n=2}^{\infty} \frac{l_n}{n!} z^n \quad g_1(z) = \sum_{n=2}^{\infty} \frac{k_n}{n!} z^n, \quad (l_0 = l_1 = k_0 = k_1 \text{ from (14)}). \tag{17}$$

Substituting (17) in the first and second of the equations (16) and equating the coefficients of different powers of z, we obtain

$$l_2 = A_1 \text{ (say)}, \quad l_3 = -2 - i\lambda, \quad l_4 = i\lambda A_1, \quad \ldots;$$

for $n \geqslant 4$, the coefficients follow a symmetrical relation,

$$k_2 = B_1 \text{ (say)}, \quad k_3 = -2c - i\lambda, \quad k_4 = i\lambda B_1, \quad \ldots;$$

for $n \geqslant 4$ the coefficients follow a symmetrical relation, where

$$f_0(z) = \sum_{n=2}^{\infty} \frac{a_n}{n!} z^n \quad \text{and} \quad g_0(z) = \sum_{n=2}^{\infty} \frac{b_n}{n!} z^n \quad (a_2 = A,\, b_2 = B);$$

a_n and b_n are given by Hayday & Bowlus (1967).

Following Meksyn, we temporarily treat the first and second equations of (16) as a linear system for f'' and g'', so

$$\left.\begin{aligned} f_1''(z) &= A_1 e^{-F(z)} + e^{-F(z)} \int_0^z e^{F(z)} \{2(f_0' f_1' - 1) + i\lambda(f_1' - 1) - (f_1 + c g_1) f_0''\}\, dz, \\ g_1''(z) &= B_1 e^{-F(z)} + e^{-F(z)} \int_0^z e^{F(z)} \{2c(g_0' g_1' - 1) + i\lambda(g_1' - 1) - (f_1 + c g_1) g_0''\}\, dz, \end{aligned}\right\} \tag{18}$$

where
$$F(z) = \int_0^z (f_0 + c g_0)\, dz. \tag{19}$$

This shows that f_1'' and g_1'' may be of the form

$$f_1'' = e^{-F} \phi_1(z) \quad \text{and} \quad g_1'' = e^{-F} \phi_2(z), \tag{20}$$

$$\phi_1(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^n, \quad \phi_2(z) = \sum_{n=0}^{\infty} \frac{\delta_n}{n!} z^n, \tag{21}$$

where $\phi_1(z)$, $\phi_2(z)$ are slowly-varying functions. Equation (20) may be regarded as the asymptotic solution of the first and second equations of (16).

γ_n is determined from the relation

$$\sum_{n=2}^{\infty} l_n \frac{n(n-1)}{n!} z^{n-2} = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^n \exp\left\{ - \sum_{n=2}^{\infty} (a_n + c b_n) \frac{1}{(n+1)!} z^{n+1} \right\};$$

expanding both the sides and comparing different powers of z we get

$$\begin{aligned} &\gamma_0 = l_2 = A_1, \quad \gamma_1 = l_3 = -2 - i\lambda, \quad \gamma_2 = l_4 = iA_1\lambda, \\ &\gamma_3 = 3AA_1 - cAB_1 - 2i\lambda + \lambda^2, \\ &\gamma_4 = [(1 + c^2)\, 2A + 3(A_1 + cB_1) - 6(2A + A_1) - \lambda^2 A_1] \\ &\qquad\qquad - (2 + i\lambda)(A + cB) + i\lambda A(5 - c); \end{aligned}$$

and similarly from (20)

$$\begin{aligned} &\delta_0 = k_2 = B_1, \quad \delta_1 = k_3 = -2c - i\lambda, \quad \delta_2 = k_4 = i\lambda B_1, \\ &\delta_3 = 3cBB_1 - A_1 B - 2ic\lambda + \lambda^2, \\ &\delta_4 = [(1 + c^2)\, 2B + 2c(A_1 + cB_1) - 12Bc^2 - 6c^2 B_1 - \lambda^2 B_1] \\ &\qquad\qquad - (2c + \lambda i)(A + cB) + Bi\lambda(1 - 5c). \end{aligned}$$

The velocity field is obtained by integrating (20) as follows:

$$f_1'(z) = \int_0^z e^{-F(z)} \phi_1(z) \quad \text{and} \quad g_1'(z) = \int_0^z e^{-F(z)} \phi_2(z)\, dz. \tag{22}$$

The integration is done by the method of steepest descent. Now let

$$F(z) = z^3 \sum_{m=0}^{\infty} c_m z^m = \tau, \tag{23}$$

whence we get

$$z = \sum_{m=0}^{\infty} [A_m/(m+1)]\tau^{\frac{1}{3}(m+1)}, \tag{24}$$

so that

$$f_1' = \int_0 e^{-\tau'}\phi_1(z)\,\frac{dz}{d\tau'}\,d\tau', \quad g_1' = \int_0 e^{-\tau'}\phi_2(z)\,\frac{dz}{d\tau'}\,d\tau', \tag{25}$$

where

$$\left.\begin{aligned} \phi_1(z)(dz/d\tau) &= \tau^{-\frac{2}{3}} \sum_{m=0}^{\infty} d_{1m}\,\tau^{\frac{1}{3}m},\\ \phi_2(z)(dz/d\tau) &= \tau^{-\frac{2}{3}} \sum_{m=0}^{\infty} d_{2m}\,\tau^{\frac{1}{3}m}.\end{aligned}\right\} \tag{26}$$

As

$$\lim_{z\to\infty} f^1(z) = 1 \quad \text{and} \quad \lim_{z\to\infty} g^1(z) = 1$$

we get

$$\sum_{m=0}^{\infty} d_{1m}\,\overline{|\tfrac{1}{3}(m+1)} = 1, \quad \sum_{m=0}^{\infty} d_{2m}\,\overline{|\tfrac{1}{3}(m+1)} = 1. \tag{27}$$

For a set of values of A, B, c, the constants A and B are determined from (27), characterizing a fluctuating flow about a mean flow (characterized by A, B and c).

For different types of the main flow (for which c has different values) A, B, are given by Howarth (1951), Hayday & Bowlus (1967).

The expressions of c_m and A_m have already been calculated by Hayday & Bowlus (1967, p. 420). From the first equation of (26) we obtain

$$\int^{(0+)} \tau^{-\frac{1}{3}(m+1)}\phi_1(z)\,dz = d_{1m}\int^{(0+,\,0+,\,0+)}\frac{d\tau}{\tau} = 6\pi i d_{1m};$$

therefore

$$d_{1m} = \frac{1}{6\pi i}\int^{(0+)} \tau^{-\frac{1}{3}(m+1)}\phi_1(z)\,dz, \tag{28}$$

so that d_{1m} is one-third the coefficient of z^{-1}. Since

$$\tau^{-\frac{1}{3}(m+1)} = z^{-(m+1)}(c_0 + c_1 z + c_2 z^2 + \ldots)^{-\frac{1}{3}(m+1)},$$

d_{1m} is one-third the coefficient of z^m in the expression

$$(c_0 + c_1 z + c_2 z^2 + \ldots)^{-\frac{1}{3}(m+1)} \sum_{n=0}^{\infty} \frac{\gamma_n z^n}{n!}. \tag{29}$$

Similarly, d_{2m} is one-third the coefficient of z^m in the expression

$$(c_0 + c_1 z + c_2 z^2 + \ldots)^{-\frac{1}{3}(m+1)} \sum_{n=0}^{\infty} \frac{\delta_n z^n}{n!}. \tag{30}$$

d_{1m} and d_{2m}, thus calculated, are expressed in terms of A, B, A_1, B_1, c.

Results

$c = 1, A = B = 1{\cdot}312$ (Howarth 1951) represents the stagnation point flow near a body of revolution. Applying Eulers transformations to (27), to calculate A_1 we get

$$A_1 = \frac{2{\cdot}394347 + 0{\cdot}791407 i\lambda - 0{\cdot}030613\lambda^2}{1{\cdot}31471 + 0{\cdot}170697 i\lambda - 0{\cdot}001555\lambda^2}. \tag{31}$$

If ω be small the intensity of skin friction

$$= \mu\left[\frac{\partial u_0}{\partial \zeta}\right]_{\zeta=0} + \epsilon\, e^{i\omega t}\mu\left[\frac{\partial u_1}{\partial \zeta}\right]_{\zeta=0} = \tau_0 + \epsilon\, e^{i\omega t}[1{\cdot}388108\tau_0 + 0{\cdot}278586(i\omega/a)\,\tau_0]. \quad (32)$$

When ω is large

Velocity distribution. For large-frequency parameter we can approximate the equations (2) and (3) according to the theory of differential equations with large parameters. This procedure was also employed by Lighthill (1954).

In (2) and (3), retaining the terms involving ω and the derivatives of highest order, we obtain

$$i(u_1 - U_1) = \partial^2 u_1/\partial\zeta^2, \quad i(v_1 - V_1) = \partial^2 v_1/\partial\zeta^2. \quad (33), (34)$$

These equations are identical to the equation for 'shear waves'. So the flow is of shear-wave type similar to that discussed by Lighthill, in the case of a two-dimensional flow. These terms dominate at higher frequencies, since viscosity does not get time to combat the velocity fluctuations imposed by the external pressure gradient fluctuations, except in the 'shear wave' layer, near the wall whose thickness if of the order $(\nu/\omega)^{\frac{1}{2}}$. The solution of the equations is

$$u_1 = U_1[1 - e^{-\zeta(i\omega/\nu)^{\frac{1}{2}}}], \quad (35)$$

$$v_1 = V_1[1 - e^{-\zeta(i\omega/\nu)^{\frac{1}{2}}}]. \quad (36)$$

Equations (35) and (36) give u_1 and v_1 for large values of ω, whatever the nature of the mainstream fluctuations. If $U_0 = U_1$, $V_0 = V_1$ then the intensity of skin friction in X and Y directions are

$$[\mu(\partial u_0/\partial\zeta)_{\zeta=0} + \epsilon\, e^{i\omega t}\mu U_0(i\omega/\nu)^{\frac{1}{2}}], \quad (37)$$

$$[\mu(\partial V_0/\partial\zeta)_{\zeta=0} + \epsilon\, e^{i\omega t}\mu V_0(i\omega/\nu)^{\frac{1}{2}}]. \quad (38)$$

Thus it is evident that the amplitude of the skin friction increases with frequency, and its phase is ahead of that of the fluctuations in the stream by 45°.

Comparing (37) with (32) we find that the phase of the skin-friction fluctuations will rise to its ultimate value 45°, when

$$\frac{\omega}{a} = \frac{1{\cdot}388108}{0{\cdot}278586} = \frac{\omega_0}{a} \quad \text{(say)}. \quad (39)$$

The amplitude is then, according to (32), $\epsilon \times 1{\cdot}388108\tau_0 \times 2^{\frac{1}{2}}$, and that according to (37) is $\epsilon\mu U_0(\omega_0/\nu)^{\frac{1}{2}}$, so that their ratio is equal to

$$\frac{1{\cdot}399108 \times (a/\nu)^{\frac{1}{2}}\, 2^{\frac{1}{2}} \times 1{\cdot}315}{(\omega_0/\nu)^{\frac{1}{2}}} = 1{\cdot}153825 \quad \text{(say } R\text{)}. \quad (40)$$

Hence both the phase and amplitude of the skin-friction fluctuations agree at frequency ω_0, so in the present case this value of ω_0 may be regarded as the boundary between the regions of applicability of (32) and (38) or (37) (because $c = 1$, they are identical).

For other values of c, namely $c = 0$, 0·25, 0·50, 0·75, calculations were carried out on the IBM-360 in the Institute of Computer Science, University of Toronto. At first, calculations were carried out without using Eulers transformation (in the case of (32) also), but it was seen that the transformation gave better results when dealing with (27). For brevity, only the results applying the transformation are given here.

A_1 and B_1 are of the form

$$A_1 = \frac{(A1)+(A2)i\lambda+(A3)\lambda^2+(A4)i\lambda^3+(A5)\lambda^4}{(A6)+(A7)i\lambda+(A8)\lambda^2+(A9)i\lambda^3}, \tag{41}$$

$$A_2 = \frac{(A10)+(A11)i\lambda+(A12)\lambda^2+(A13)i\lambda^3+(A14)\lambda^4}{(A6)+(A7)i\lambda+(A8)\lambda^2+(A9)i\lambda^3}. \tag{42}$$

The values of $A1, A2, \ldots, A14$ are given in tables 1 and 2. Table 1 gives the values based on Howarth's (1951) results and table 2 gives those based on Hayday & Bowlus (1967).

c	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$	$A7$
0	5·868252	3·639445	−0·666450	−0·037355	0·000460	3·173846	1·248844
0·25	5·122398	2·978586	−0·501689	−0·025611	0·000281	2·768908	1·012312
0·50	4·429901	2·360212	−0·355571	−0·016081	0·000154	2·388988	0·779737
0·75	3·864244	1·878446	−0·231364	−0·010014	0·000084	2·075738	0·596729

c	$A8$	$A9$	$A10$	$A11$	$A12$	$A13$	$A14$
0	−0·003825	0·000030	2·094218	2·838103	−0·433781	−0·036230	0·000460
0·25	−0·002542	0·000018	2·893636	2·534364	−0·303769	−0·025090	0·000281
0·50	−0·001516	0·000009	3·261739	2·150840	−0·204574	−0·015890	0·000154
0·75	−0·000890	0·000005	3·396544	1·804703	−0·137861	−0·009962	0·000084

c	$B1$	$B2$	$B3$	$B4$	$B5$	$B6$	$B7$
0	−0·895697	−0·381715	0·048833	0·008983	0·010139	0·317962	5·046827
0·25	−0·822637	−0·344854	0·024720	0·006694	0·007407	0·286235	4·182424
0·50	−0·729227	−0·276132	0·009172	0·004162	0·004551	0·229038	3·397985
0·75	−0·641658	−0·214409	0·001955	0·002480	0·002693	0·177812	2·773909

c	$B8$	$B9$	$B10$	$B11$	$B12$	$B13$
0	6·222554	−1·110994	0·212721	0·005154	−0·000630	0·000005
0·25	4·444373	−0·694814	0·131801	0·002703	−0·000322	0·000002
0·50	3·082338	−0·414061	0·072910	0·001265	−0·000139	0·000001
0·75	2·172920	−0·252675	0·039848	0·000596	−0·000058	0·000000

TABLE 1. Calculations based on the values of A and B as given by Howarth (1951)

When $c \to 0$, as discussed at the beginning of the present paper, we obtain, when ω is small, τ_{x0} = skin friction chordwise

$$= \tau_A + \epsilon\, e^{i\omega t}[1{\cdot}499546\tau_A + 0{\cdot}339966(i\omega/a)\,\tau_A]. \tag{43}$$

This gives the skin friction on the circular cylinder near the stagnation point when the axis is inclined to the mainstream at an angle V_0/U_0.

As in the case $c = 1$, Lighthill's analysis may be extended to the cases $c = 0{\cdot}0$, $0{\cdot}25$, $0{\cdot}50$, $0{\cdot}75$, both chordwise and spanwise. For small ω, the chordwise intensity of skin friction may be given as follows:

$$\tau_{x0} = \tau_A + \epsilon\, e^{i\omega t}[D_1 \tau_A + D_2 \tau_A (i\omega/a)], \tag{44}$$

where τ_A, D_1, D_2 are given in table 3. For large ω they are given by (37).

c	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$	$A7$
0	5·901954	3·665812	−0·672436	−0·037784	0·000468	3·189692	1·256896
0·25	5·103109	2·962019	−0·497853	−0·025354	0·000277	2·758177	1·006680
0·50	4·417043	2·349984	−0·355440	−0·015954	0·000152	2·381783	0·776306
0·75	3·852838	1·870483	−0·249924	−0·009939	0·000083	2·069717	0·594215

c	$A8$	$A9$	$A10$	$A11$	$A12$	$A13$	$A14$
0	−0·003871	0·000030	2·104826	2·858665	−0·432705	−0·036642	0·000486
0·25	−0·002515	0·000017	2·887812	2·521001	−0·294452	−0·024840	0·000277
0·50	−0·001503	0·000009	3·252269	2·141143	−0·201034	−0·015765	0·000152
0·75	−0·000882	0·000005	3·386218	1·796873	−0·136290	−0·009888	0·000083

c	$B1$	$B2$	$B3$	$B4$	$B5$	$B6$	$B7$
0	−0·907202	−0·387626	0·049825	0·009151	0·010330	0·332158	5·086876
0·25	−0·815204	−0·340779	0·023867	0·006586	0·007286	0·283087	4·159394
0·50	−0·723890	−0·273532	0·008799	0·004109	0·004493	0·227060	3·382920
0·75	−0·636978	−0·212447	0·001808	0·002450	0·002660	0·176343	2·761657

c	$B8$	$B9$	$B10$	$B11$	$B12$	$B13$
0	6·298179	−1·128517	0·215837	0·005260	−0·000643	0·000005
0·25	4·402022	−0·685645	0·130129	0·002656	−0·000317	0·000002
0·50	3·852838	−0·409856	0·072180	0·001248	−0·000137	0·000001
0·75	2·157632	−0·250255	0·039466	0·000588	−0·000057	0·000000

TABLE 2. Calculation based on the values of A and B as given by Hayday & Bowlus (1967)

c	τ_A	D_1	D_2	R
0·00	1·233	1·499546	0·339966	1·245018
0·25	1·247	1·483537	0·320270	1·215595
0·50	1·267	1·463536	0·302078	1·191387
0·75	1·288	1·445260	0·287094	1·173360
1·00	1·312	1·388108	0·278586	1·153825

TABLE 3. The τ_A's are given by Howarth (1957)

Comparing (44) with (37), we find that the phase of the skin-friction fluctuations will rise to its ultimate value 45°, when

$$\frac{\omega}{a} = \frac{D_1}{D_2} = \frac{\omega_{0x}}{a}. \tag{45}$$

Then let the ratio of the amplitudes of skin friction as given by (44) and (37), be R. For each c the value of R is calculated and given in table 3.

From the values of R given in table 3 it is evident that both the phase and amplitude of skin-friction fluctuations agree at frequency ω_{0x}. So as in the case $c = 1$ this value of ω_{0x} may be regarded as the boundary between the regions of applicability of (44) and (37).

It also may be noted from table 3 that as $c \to 1$ (from 0·0) the body becomes more and more symmetrical, and the quasi-steady and unsteady intensity of skin friction continue to diminish. They are minimum when the body is a solid of revolution.

When ω is small the spanwise skin friction is given by the following formula

$$\tau_y = \tau_B + \epsilon\, e^{i\omega t}[E_1 \tau_B + E_2 \tau_B (i\omega/a)], \tag{46}$$

where τ_B, E_1, E_2, are given in table 4. For large ω they are given by (38).

Lighthill's analysis may also be applied to spanwise skin frictions.

From (46) and (38) it is evident that the phase of the skin-friction fluctuations will rise to its ultimate value 45°, when

$$\frac{\omega}{a} = \frac{E_1}{E_2} = \frac{\omega_{0y}}{a}. \tag{47}$$

Let R be the ratio of the amplitudes of fluctuating skin friction as given by (46) and (38). For different c the values of R are given in table 4.

c	τ_B	E_1	E_2	R
0·00	0·570	1·157607	1·113305	0·915119
0·25	0·805	1·298194	0·662391	1·055694
0·50	0·998	1·368059	0·508832	1·77565
0·75	1·164	1·375762	0·342805	1·142740
1·00	1·312	1·388108	0·278586	1·153825

TABLE 4. τ_B's are given by Howarth (1951).

From the values of R given in table 4 it is clear that both the phase and amplitude of skin-friction fluctuations agree at frequency ω_{0y} (different c), so that this value of ω_{0y} may be regarded as the boundary between the regions of applicability of (46) and (38).

Therefore, as in the case of chordwise friction, for each point on the body there is a critical frequency ω_{0y} such that for frequencies $\omega > \omega_{0y}$, the oscillations are to a close approximation ordinary 'shear waves', unaffected by the main flow; the phase advance in the skin friction is then 45°. For frequencies $\omega < \omega_{0y}$ the oscillations may be closely approximated as the sum of two parts: one quasi-steady part and the other proportional to the acceleration of the oncoming stream. The phase advance in the skin friction is then $\tan^{-1}(\omega/\omega_{0y})$.

It may be noted that as c increases both the quasi-steady and unsteady parts of the skin friction increase and ultimately become equal to the corresponding parts of chordwise frictions when $c = 1$, i.e. the body becomes a solid of revolution.

4. Heat transfer

Heat transfer may also be obtained as before.

Integrating the energy equation in (16) we obtain

$$\theta_1' = \theta_1'(0)\,e^{-P_rF(z)} + e^{-P_rF(z)}\int_0^z e^{P_rF(z)}[P_r(i\omega/a)\,\theta_1 - P_r(f_1+cg_1)\,\theta_0]\,dz,$$

which shows that θ_1' will be of the form

$$\theta_1' = e^{-P_rF(z)}\phi_3(z), \quad \text{where} \quad \phi_3(z) = \sum_{n=0}^{\infty}(L_n Z^n/n!); \tag{48}$$

$\phi_3(z)$ is a slowly-varying function.

Equation (34) may be regarded as the asymptotic solution of the energy equation in (16).

Let

$$\theta_1 = \sum_{n=0}^{\infty}\frac{h_n}{n!}z^n,$$

as before, substituting in the third equation of (16) and equating different powers of z, we obtain h_n in terms of h_1, A_1, B_1 and I_n, namely

$$h_0 = 0, \quad h_1 = (\text{unknown}), \quad h_2 = 0, \quad h_3 = P_r h_1 i\lambda,$$
$$h_4 = -P_r[h_1(A+cB) + I_1(A_1+cB_1)], \quad \text{etc.}$$

where

$$\theta_0 = \sum_{n=0}^{\infty}\frac{I_n}{n!}z^n, \quad I_0 = 1, \quad I_1 = \theta_0'(0), \quad I_2 = I_3 = 0, \quad I_4 = -IP_r(A+cB).$$

Now, the L_n's are determined from,

$$\sum_{n=1}^{\infty}\frac{nh_n}{n!}z^{n-1} = \sum_{n=0}^{\infty}\frac{L_n}{n!}z^n \exp\left\{-P_r\sum_{n=2}^{\infty}\frac{a_n+cb_n}{(n+1)!}z^{n+1}\right\},$$

so that

$$L_0 = h_1, \quad L_1 = h_2 = 0, \quad L_2 = h_3 = P_r(i\lambda)\,h_1, \quad L_3 = -P_r\,I_1(A_1+cB_1),$$
$$L_4 = P_r\,I_1\{(2+i\lambda)+c(2c+i\lambda)\} - P_r^2\lambda^2 h_1, \quad \text{etc.}$$

Integrating (34),
$$\theta_1(z) = \int_0^z e^{-P_rF(z)}\,\phi_3(z)\,dz. \tag{49}$$

Now,
$$\phi_3(z)\frac{dz}{d\tau} = \tau^{-\frac{2}{3}}\sum_{n=0}^{\infty}H_m\tau^{\frac{1}{3}m}.$$

As $z\to\infty$, $\theta_1\to 0$, so that we have

$$0 = \sum_{m=0}^{\infty}P^{\frac{1}{3}m+1}H_m\,\Gamma\!\left(\tfrac{1}{3}(m+1)\right). \tag{50}$$

As before, it is clear that $3H_m$ is the coefficient of z^m, in the following product:

$$(c_0+c_1z+c_2z^2+\ldots)^{-\frac{1}{3}(m+1)}\sum_{n=0}^{\infty}\frac{L_nz^n}{n!}.$$

Thus, H_m are expressed in terms of A, B, c, A_1, B_1.

Results

When $c = 1$. $A = B = 1{\cdot}312$ and $P_r = 1$,

$$h_1 = \frac{-0{\cdot}574239 - 0{\cdot}130366 i\lambda - 0{\cdot}003246\lambda^2 - 0{\cdot}000117 i\lambda^3}{1{\cdot}728004 + 0{\cdot}938488 i\lambda - 0{\cdot}155183\lambda^2 - 0{\cdot}008689 i\lambda^3}. \tag{51}$$

If ω is small, the heat transfer per unit area

$$= h = h_0 + \epsilon e^{i\omega t}[0{\cdot}437888 h_0 - (i\omega/a) \times 0{\cdot}138408 \times h_0]. \tag{52}$$

For values of c, namely 0·0, 0·25, 0·50, 0·75, h_1 may be represented as follows (the corresponding values of $\theta_0^1(0)$ are $-0{\cdot}5718$, $-0{\cdot}6172$, $-0{\cdot}6639$, $-0{\cdot}7118$ and $-0{\cdot}7589$ given by Hayday and Bowlus (1967))

$$h_1 = \frac{(B1) + i\lambda(B2) + \lambda^2(B3) + i\lambda^3(B4) + \lambda^4(B5) + i\lambda^5(B6)}{(B7) + i\lambda(B8) + \lambda^2(B9) + i\lambda^3(B10) + \lambda^4(B11) + i\lambda^5(B12) + \lambda^6(B13)}. \tag{53}$$

The values of $B1$, $B2$, $B3$, ..., $B13$, are given in tables 1 and 2. When ω is small, the intensity of heat transfer is given by

$$h = h_0 + \epsilon e^{i\omega t}[H_1 h_0 + (i\omega/a) H_2 h_0],$$

where h_0, H_1, H_2 are given in table 5.

c	h_0	H_1	H_2
0·00	0·5718	0·310383	−0·250416
0·25	0·6172	0·318680	−0·205047
0·50	0·6639	0·323250	−0·170819
0·75	0·7118	0·324978	−0·145978
1·00	0·7589	0·437888	−0·138408

TABLE 5

When ω is large

For large-frequency parameter the dominant terms of equation (4) are as follows (see Lighthill 1954):

$$\frac{\nu}{P_r}\frac{\partial^2\theta_1}{\partial\zeta^2} - i\omega\theta_1 = W_1\frac{\partial\theta_0}{\partial\zeta},$$

where $$W = W_0 + W_1 \epsilon e^{i\omega t}, \tag{54}$$

so that from the equation of continuity and (35) and (36) we obtain

$$W_1 = -(a+b)\left[\zeta - \frac{1 - e^{-\zeta(i\omega/\nu)^{\frac{1}{2}}}}{(i\omega/\nu)^{\frac{1}{2}}}\right]. \tag{55}$$

For large ω the solution of (54) may be given by (Lighthill 1954):

$$\theta_1 = (i/\omega)\, W_1\, (\partial\theta_0/\partial\zeta). \tag{56}$$

Equation (56) is valid away from the wall where u_1 and v_1 are not changing rapidly.

For small y the exponentials in u_1 and v_1 make them change too rapidly, so that the approximate solution in (56) is not valid there. But we can represent θ_0 in the form $1+\zeta[\partial\theta_0/\partial\zeta]_{\zeta=0}$ on the right side of (54). This is a fair approximation near the wall since from (4) we obtain $[\partial^2 T/\partial y^2]_{y=0} = 0$.

Under this approximation we obtain from (54)

$$\theta_1 = -\frac{i}{\omega}(a+b)\left[\frac{\partial\theta_0}{\partial\zeta}\right]_{\zeta=0}\left[\zeta-\frac{1-P_r+P_r e^{-\zeta(i\omega/\nu)^{\frac{1}{2}}}-e^{-(i\omega/\nu)^{\frac{1}{2}}}}{(1-P_r)\sqrt{(i\omega/\nu)}}\right],$$

where $P_r = \nu/k$ is the Prandtl number. The heat-transfer rate per unit area is

$$-\Theta\left[\frac{\partial\theta_0}{\partial\zeta}\right]_{\zeta=0}+\frac{i}{\omega}\epsilon\, e^{i\omega t}\kappa(a+b)\left[\frac{\partial\theta_0}{\partial\zeta}\right]_{\zeta=0}\times\frac{1}{1+(P_r)^{\frac{1}{2}}};\quad \Theta = T_\omega - T_\infty,$$

which shows that the amplitude decreases with frequency and its phase is behind that of the mainstream fluctuations by 90°. These results differs from the results of the skin friction.

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